Invariant spectra of orbits in dynamical systems

1994 J. Phys. A: Math. Gen. 27 4899

(http://iopscience.iop.org/0305-4470/27/14/017)

J. Phys. A: Math. Gen. 27 (1994) 4899–4909. Printed in the UK

Invariant spectra of orbits in dynamical systems

N Voglis† and G J Contopoulos†‡

† Department of Astronomy, University of Athens, GR-157 84, Athens, Greece
‡ Department of Astronomy, University of Florida, Gainsville, FL 32611, USA

Received 14 March 1994

Abstract. We show that in deterministic dynamical systems any orbit is associated with an invariant spectrum of stretching numbers, i.e. numbers expressing the logarithmic divergences of neighbouring orbits within one period. The first moment of this invariant spectrum is the maximal Lyapunov characteristic number (LCN). In the case of a chaotic domain, a single invariant spectrum characterizes the whole domain. The invariance of this spectrum allows the estimation of the LCN by calculating, for short times, many orbits with initial conditions in the same chaotic region instead of calculating one orbit for extremely long times. However, if part of the initial conditions are in an ordered region, the average of the short-time calculations may deviate considerably from the LCN. Invariant spectra appear not only for conservative but also for dissipative systems. A few examples are given.

1. Introduction

In deterministic dynamical systems, chaotic motion is characterized by a great sensitivity on small variations of initial conditions. Nearby chaotic orbits diverge exponentially in time t. In contrast, regular orbits diverge algebraically in time t. One way to distinguish between these two cases is to calculate the maximal Lyapunov characteristic number (LCN) defined by

$$\mathrm{LCN} = \lim \frac{1}{t} \ln \frac{\xi(t)}{\xi(0)} \qquad \text{as } t \to \infty \tag{1}$$

where $\xi(t)$ is the distance between neighbouring orbits, being initially $\xi(0)$. In practice, $\xi(t)$ is a solution of the variational equations, calculated together with the orbit itself [1]. If $\xi(t)$ grows exponentially in time then $\ln(\xi(t)/\xi(0))$ grows linearly in t and the above limit in equation (1) tends to a finite positive number which is the same for almost all directions of $\xi(0)$. If $\xi(t)$ grows as a power law, say t^p, the limit in equation (1) is zero for any value of p. Thus, an orbit can be characterized by its LCN. In practice, the numerical evaluation of LCN is not a trivial matter. The difficulty arises from the fact that, for a reliable value of LCN, an orbit has to be followed for an extremely long time (of the order of millions of periods). In most real dynamical systems however, such a time scale is too long to be realistic, e.g. in the case of galaxies, time scales larger than about a hundred periods exceed the age of the Universe itself. A reasonable question, therefore, is whether there is any alternative process to estimate the LCN of a chaotic domain that avoids the long period of computation [2, 3]. In this paper we show that it is possible to obtain accurate values of the LCN for a chaotic domain by calculating, for a short time, many orbits starting from various initial conditions provided that these initial conditions belong to the same chaotic domain.

0305-4470/94/144899+11$19.50

However, if part of the initial conditions are on the ordered region, the average LCN may deviate considerably from the LCN of the chaotic region.

In section 2 we define two quantities, i.e. the stretching number and the spectrum of stretching numbers of an orbit. In section 3, we show that the spectrum of stretching numbers is invariant for a particular orbit and also for all orbits in the same chaotic domain. It is this invariant property that allows us to estimate the LCN from many chaotic orbits calculated for a short period.

2. The stretching number

Consider an area-preserving two-dimensional map

$$x' = f(x, y, K) \qquad y' = g(x, y, K) \qquad \text{mod } 1 \tag{2}$$

where K is a nonlinearity parameter. The tangent map is

$$\mathrm{d}x' = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y \tag{3a}$$

$$\mathrm{d}y' = \frac{\partial g}{\partial x}\,\mathrm{d}x + \frac{\partial g}{\partial y}\,\mathrm{d}y. \tag{3b}$$

Two neighbouring points (x, y) and $(x + \mathrm{d}x, y + \mathrm{d}y)$ define a line element $\mathrm{d}s$ given by

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 \tag{4a}$$

with a slope

$$y_x = \mathrm{d}y/\mathrm{d}x. \tag{4b}$$

After one period this line element is mapped to $\mathrm{d}s'$ given by

$$\mathrm{d}s'^2 = \mathrm{d}x'^2 + \mathrm{d}y'^2 \tag{5a}$$

and has a slope

$$y'_{x'} = \left(\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y}y_x\right) \Big/ \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y_x\right). \tag{5b}$$

We define the stretching number a' as the quantity

$$a' = \ln\left|\frac{\mathrm{d}s'}{\mathrm{d}s}\right|. \tag{6}$$

For the adopted map

$$a' = \ln\left|\frac{\mathrm{d}s'}{\mathrm{d}s}\right| = \frac{1}{2}\ln\left[\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y_x\right)^2 + \left(\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y}y_x\right)^2 \Big/ (1 + y_x^2)\right]. \tag{7}$$

The LCN (1) in this case can be written:

$$\mathrm{LCN} = \lim \frac{1}{N} \sum_{i=1}^{N} a_i \qquad N \to \infty. \tag{8}$$

Starting from an initial point (x_0, y_0, y_{x0}), we can create a long sequence of triplets (x_i, y_i, y_{xi}) and the associated sequence of a_i. Let $\mathrm{d}N(a)$ be the number of times that the values of a_i appear in the interval $(a, a + \mathrm{d}a)$. We define the spectrum of stretching numbers as the distribution

$$S(a, K, x_0, y_0, y_{x0}) = \lim(\mathrm{d}N(a)/N\,\mathrm{d}a) \qquad N \to \infty. \tag{9}$$

Then the LCN can be written as

$$\mathrm{LCN} = \int_{-\infty}^{+\infty} S(a, K, x_0, y_0, y_{x0}) a\,\mathrm{d}a \qquad N \to \infty. \tag{10}$$

i.e. LCN is the average value of the stretching numbers a.

3. A new invariant

We have investigated numerically the behaviour of the function $S(a, K)$ in the standard map

$$y' = y + \frac{K}{2\pi} \sin 2\pi x \qquad x' = x + y' \qquad \mathrm{mod}\ 1 \tag{11}$$

and we found:

(a) that this function is independent of the initial point along any particular orbit; and

(b) that in a chaotic domain this function is independent of the initial conditions, i.e., independent of x_0, y_0, y_{x0}.

First, we found that any orbit, for a given value of K, is characterized by a unique spectrum $S(a, K)$ which is invariant for this orbit. We checked this as follows. We plotted the function $S1(a, K)$ for an orbit as it comes out from the first 10^6 periods and then plotted the function $S2(a, K)$ as it comes out from the next 10^6 periods. We found that the two resulting curves are almost identical.

Such an example is shown in figure 1(a). The full curve gives the spectrum $S1(a, K)$ from the first 10^6 periods and the dots give the spectrum $S2(a, K)$ from the next 10^6 periods. The adopted width of the bins of a is $\delta a = 0.001$. The initial conditions for the corresponding orbit are $x_0 = 0.1$, $y_0 = 0.5$ and $y_{x0} = 0.0$. The value of K used for the results of this figure is $K = 0.5$. This is a regular orbit. The same initial conditions give a chaotic orbit for $K = 5.0$ (figure 1(b)). The spectrum $S(a, K)$ acquires a new shape which is again independent of the initial point on the orbit. The same is true for any orbit that we calculated, either chaotic or ordered. The spectrum is independent of the initial value of the slope y_{x0} and of the Riemannian metric used to define $\mathrm{d}s$.

If we calculate an orbit for smaller time intervals we find a dispersion of the successive points around the limiting curve $S(a, K)$. This is seen in figure 2 where we compare the spectrum $S(a, K)$, defined from 10^5 periods, with that defined from 10^6 periods. In general, as the number of periods increases the dispersion is smaller.

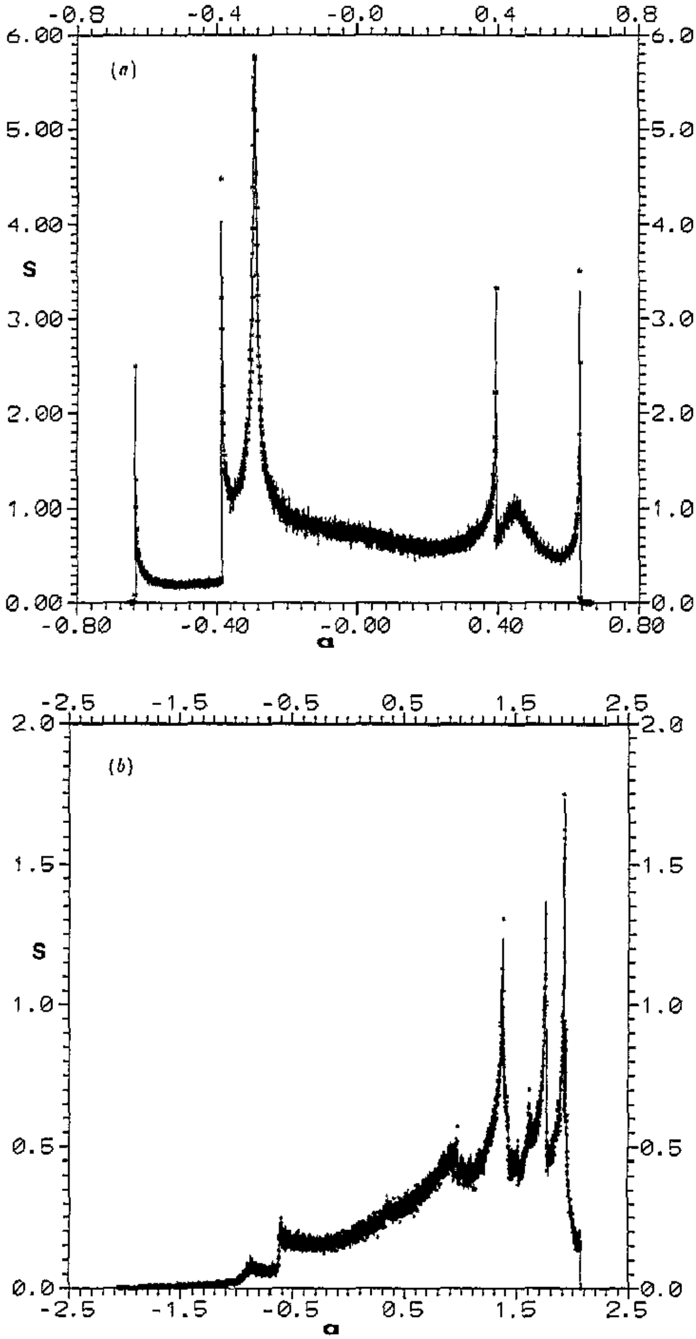


Figure 1. (*a*) The invariant spectrum of stretching numbers of an orbit in the standard map with $K = 0.5$. The initial point is $x_0 = 0.1$, $y_0 = 0.5$ with $y_{x0} = 0.0$. This is a regular orbit. The full curve gives the spectrum of the first 10^6 periods and the dots give the spectrum of the next 10^6 periods. It is obvious that the spectrum is independent of the initial point of the orbit. (*b*) The full curve gives the spectrum $S(a, K)$ of the first 10^6 periods of the orbit starting at the same initial conditions as the orbit in (*a*) but with $K = 5.0$. This is a chaotic orbit having a new invariant spectrum. The dots give the spectrum resulting from the next 10^6 periods. The two spectra practically coincide.

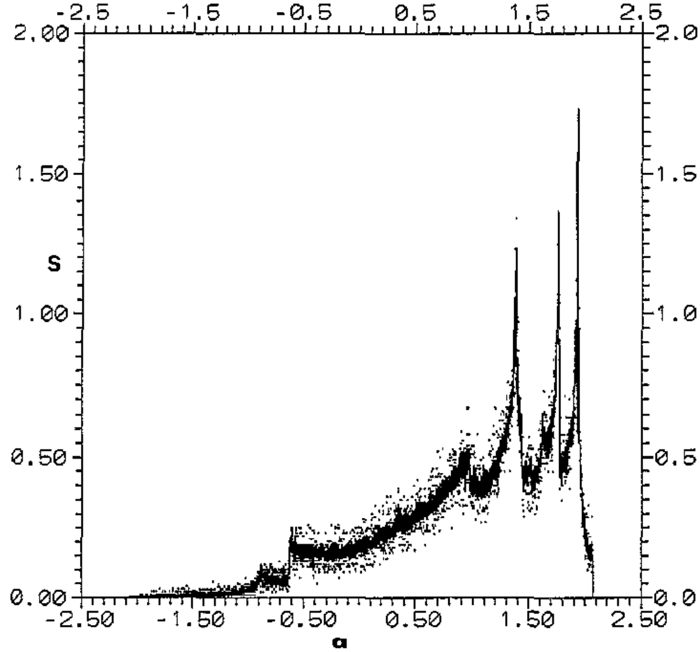


Figure 2. The full curve is the same as in figure 1(b). The dots give the spectrum $S(a, K)$ as it comes out from the first 10^5 periods only and show an appreciable dispersion. The dispersion decreases as the number of periods increases. The spectrum tends to an invariant shape $S(a, K)$ as the number of periods tends to infinity. (In practice, to 10^6 periods or more.)

Then we found that the spectrum is independent of the initial conditions, provided that they belong to the same chaotic domain of the map.

An example is shown in figure 3 where the full curve is the same as in figure 1(b) and the dots give the function $S(a, K)$ if we repeat the previous mapping with $y_{x0} = -1.0$. A broken curve corresponding to $y_{x0} = 1.0$ is also drawn in this figure but cannot be distinguished since it almost coincides with the full curve. It is clear from this plot that the function $S(a, K)$ is independent of the value of y_{x0}.

We have repeated the mapping for a series of initial conditions varying either x_0 or y_0 but taking care that their values are not inside islands of organized motion. The results show that the spectrum of stretching numbers, i.e. the function $S(a, K)$, is independent of the initial conditions if the initial conditions belong to the same chaotic domain of the map.

Extensive numerical investigations show that the function $S(a, K)$ within the limits of the available accuracy is invariant with respect to the initial conditions throughout the whole chaotic domain in the space (x, y, y_x). In the limit of infinite periods this function represents an invariant spectrum of streching numbers.

Thus the invariant spectrum of stretching numbers becomes a very important quantity in the case of chaotic orbits. A chaotic orbit passes from the neighbourhood of any point belonging to the chaotic domain. In other words, all the points of a chaotic domain are close *to the same single orbit. Therefore, if we adopt any of these points as a starting point* together with its tangent map we find the same spectrum $S(a, K)$.

A direct application of the invariant property of $S(a, K)$ in chaotic domains is the following. Instead of calculating one orbit for millions of periods in order to obtain $S(a, K)$, and hence, the LCN, we can calculate, for a short time, many orbits starting from different

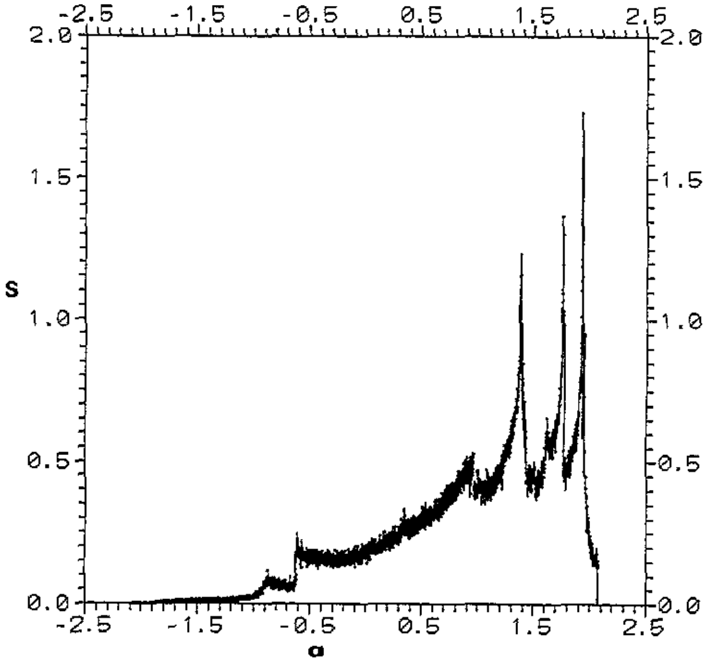


Figure 3. The full curve is the same as in figure 1(*b*). The dots give the spectrum of stretching numbers if the initial point is the same but the initial slope of the tangent map is $y_{x0} = -1.0$. In the same figure a broken curve corresponding to the same x_0, y_0 and $y_{x0} = 1.0$ has been drawn but cannot be distinguished because it coincides with the full curve.

initial points in the chaotic domain. The two calculations lead to the same invariant spectrum $S(a, K)$ and, therefore, the same Lyapunov number.

Such an example is given in figure 4 where the full curve gives the spectrum of the reference orbit ($x_0 = 0.1$, $y_0 = 0.5$, $y_{x0} = 0.0$ for $K = 5.0$) shown in figure 1(*b*) and the dots give the function $S(a, K)$ obtained by calculating, for only 100 periods, the evolution of 100×100 initial points located at the nodes of a 100×100 square grid covering an area 0.1×0.1 of the unit square. For every one of these initial points, the initial slope of the respective tangent map is taken to be equal to zero. We call this set of initial points the 'sampling box'. The first point of this sampling box is exactly the first initial point of the reference orbit ($x_0 = 0.1$, $y_0 = 0.5$, $y_{x0} = 0.0$, $K = 5.0$). The box is entirely in the chaotic region. It is clear from this figure that this box reproduces the spectrum $S(a, K)$.

If we choose another position for the sampling box, provided that the whole box is again inside the chaotic region, the result is the same. The box reproduces the same spectrum $S(a, K)$.

However, if the location of the sampling box is chosen inside an island of organized motion the results are completely different. Such an example is shown in figure 5. In this figure, the spectrum of a sampling box starting at $x_0 = 0.64$, $y_0 = 0.32$ is plotted together with the spectrum of the orbit ($x_0 = 0.1$, $y_0 = 0.5$, $y_{x0} = 0.0$ for $K = 5.0$). These two spectra are now completely different.

The spectrum resulting from the last box is not a single spectrum. It is a superposition of many spectra corresponding to various orbits in the ordered region. If the position of the sampling box changes, the corresponding mixture of spectra changes because new types of

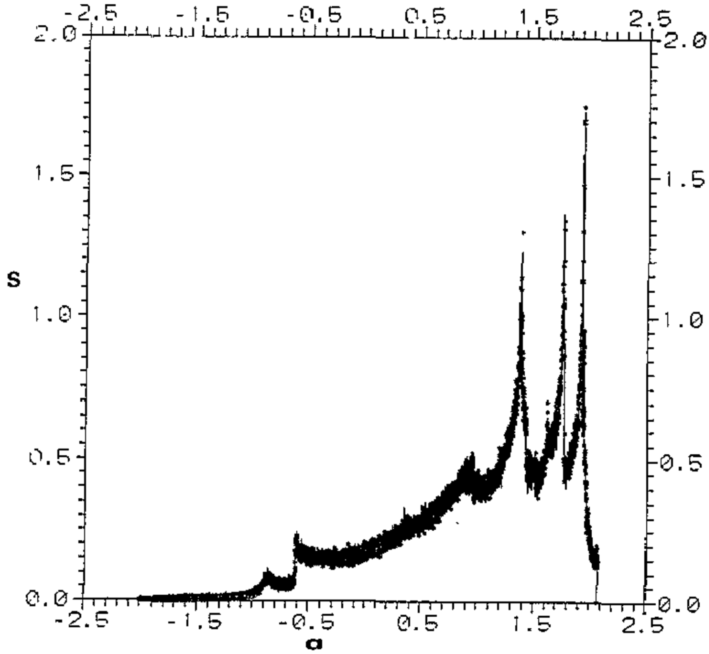


Figure 4. The full curve is the same as in figure 1(*b*). The dots give the spectrum resulting from calculating, for only 100 periods, 100 × 100 orbits having initial points in a sampling box belonging to the chaotic domain. This calculation reproduces the same invariant spectrum.

orbits enter the box. This happens until the whole box enters the chaotic domain where the spectrum takes a fixed form.

However, it is possible that a given sampling box is not completely in the chaotic domain. There is always a possibility that a given box contains some ordered regions. If this is the case, the calculated spectrum is again an average between chaotic and ordered spectra and the limit is not well defined. Therefore, it is necessary to have a preliminary investigation to check whether all the orbits in the sampling box are chaotic or not.

The same is true with regard to the Lyapunov numbers. We can define an 'average Lyapunov number' (ALN) over a short interval of time, as Udry and Pfenniger [2] did for galactic orbits. However, unless we know that all the orbits are chaotic, this ALN is not equal to the Lyapunov number defined as the limit (1) for individual orbits in the chaotic region. It is only hoped that if the portion of the regular orbits in the sampling box is very small, their effect in determining the invariant spectrum of the chaotic region may not be important and the error in the ALN may be small.

Kandrup and Mahon [3] have emphasized the need to select the various orbits in the chaotic region. The Lyapunov numbers for all such orbits are the same and they are equal to the ALN calculated for short time intervals for a large number of initial conditions.

One remark should apply to cases of small K in which there may be several chaotic regions separated by closed KAM curves. In such cases, the spectra and LCNs in different chaotic regions are, in general, different. Therefore our arguments about a unique spectrum and consequently a unique Lyapunov number for all chaotic orbits refer to orbits in connected chaotic regions that are not separated from each other by KAM curves.

We conclude by stressing the fact that we found not only an invariant Lyapunov number

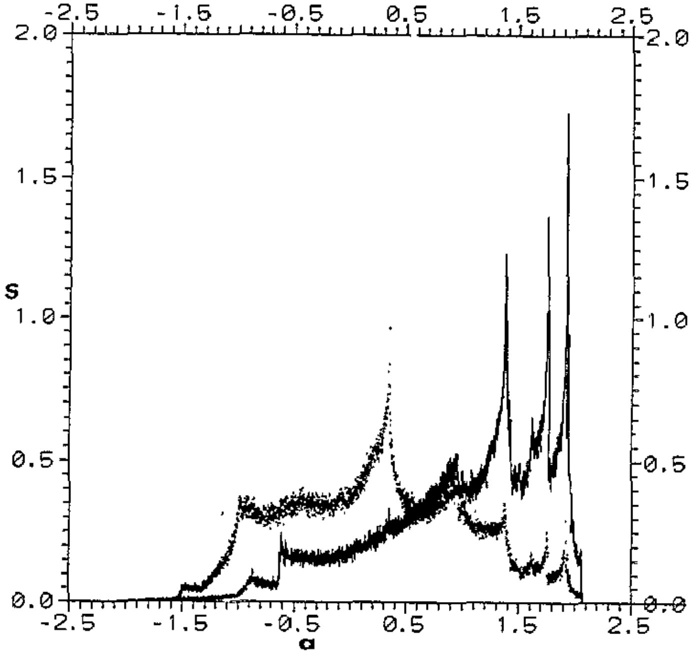


Figure 5. The full curve is the same as in figure 1(b). The dots give a spectrum resulting from the sampling box if it is located inside an island of organized motion. This position of the box does not reproduce the same spectrum. It gives a mixture of spectra of various orbits. If the box moves towards the chaotic domain the corresponding mixture of spectra moves towards the fixed form given by the full curve.

but a much more detailed entity in the chaotic region, namely an invariant spectrum. It is remarkable that this spectrum, which has several peculiarities (maxima and minima), is exactly reproduced for all the orbits of the same chaotic region.

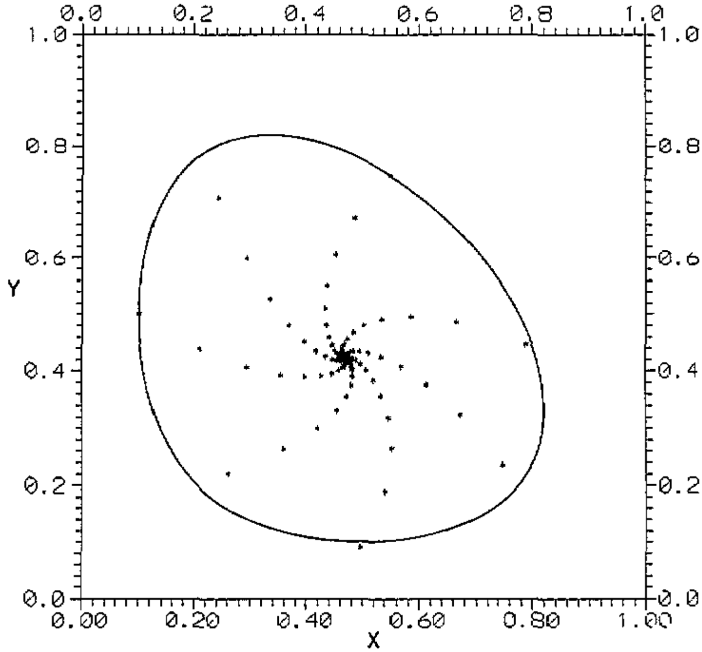
Of course, the invariant spectrum depends on the particular mapping chosen. We have checked that the invariant spectrum is different for other mappings. However, the existence of invariant spectra for the chaotic regions of dynamical systems seems to be a generic phenomenon.

4. Invariant spectra in dissipative systems

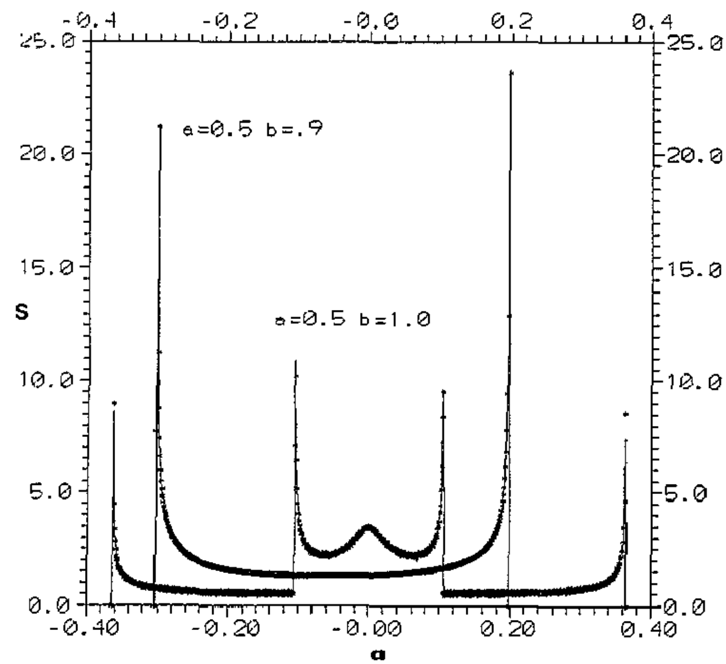
We have found that invariant spectra exist not only in conservative systems but also in dissipative systems. We have checked this by using the Hénon map [4]

$$x' = 1 - Kx^2 - y \qquad y' = bx \qquad \mathrm{mod}\ 1. \tag{12}$$

As is well known, this map is non-dissipative for $b = 1.0$ and dissipative for $b < 1.0$. In figure 6(a), the full curve gives a regular orbit in the non-dissipative Hénon map ($K = 0.5$ and $b = 1.0$) and the dots give a regular orbit in the dissipative map ($K = 0.5$ and $b = 0.9$). Both orbits start from the same initial point $x_0 = 0.1$, $y_0 = 0.5$. The corresponding invariant spectra are shown in figure 6(b). The non-dissipative case is the spectrum with the five maxima (four sharp maxima and one smooth maximum near the value of $a = 0.0$). The



(*a*)



(*b*)

Figure 6. (*a*) A non-dissipative regular orbit (full curve) of the Hénon map ($K = 0.5$, $b = 1.0$) and a dissipative orbit ($K = 0.5$, $b = 0.9$) starting from the same initial point ($x_0 = 0.1, y_0 = 0.5$). (*b*) The invariant spectra of the orbits of (*a*). The spectrum with the five maxima (four sharp and one smooth in the middle) belongs to the non-dissipative orbit. The other (U-shaped) belongs to the dissipative orbit.

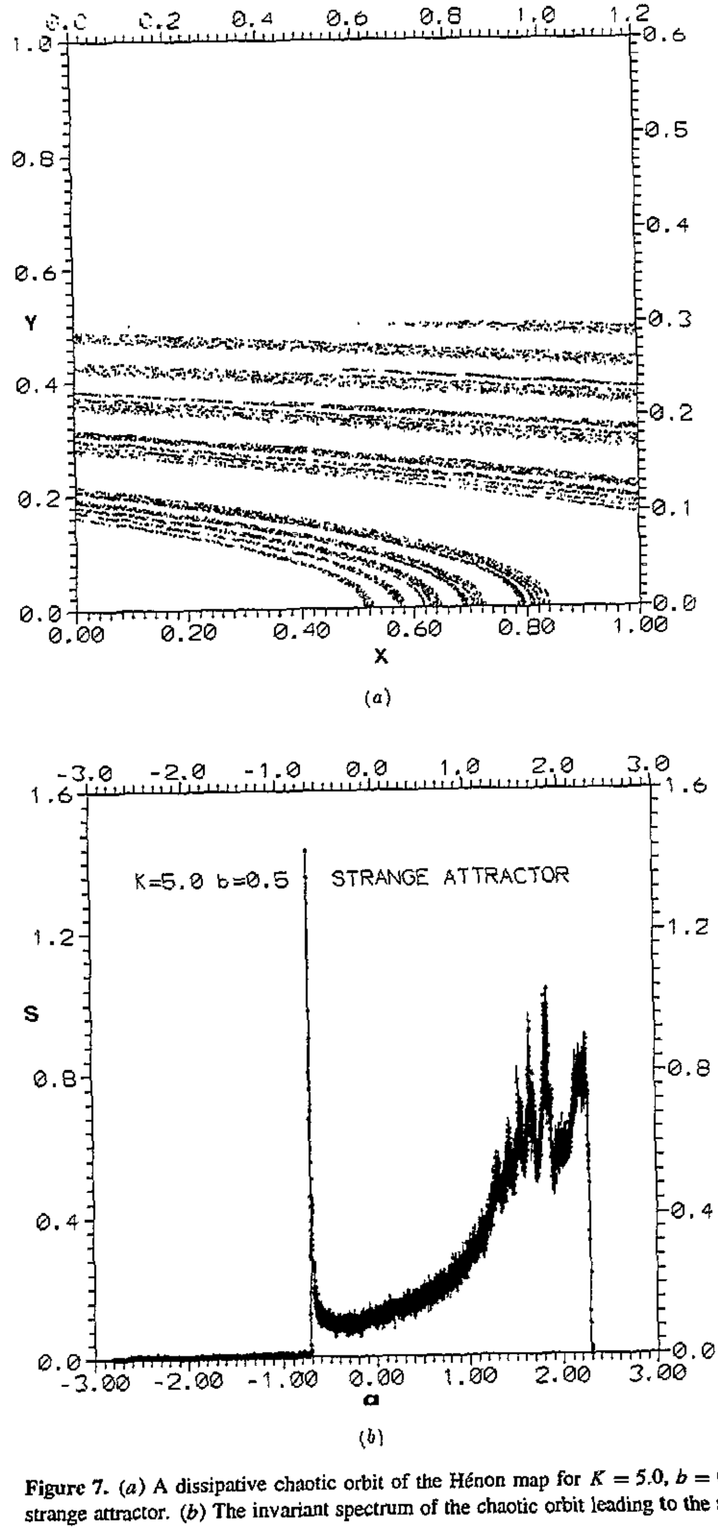


Figure 7. (*a*) A dissipative chaotic orbit of the Hénon map for $K = 5.0$, $b = 0.5$ leading to a strange attractor. (*b*) The invariant spectrum of the chaotic orbit leading to the strange attractor shown in (*a*).

dissipative case is the spectrum with only two sharp maxima at the ends. Again the full curves give the spectrum of the first 10^6 periods while the dots give the spectrum of the next 10^6 periods. It is clear that the spectra of stretching numbers are invariant even in dissipative systems.

In the dissipative case most of the points of the orbit in figure 6(a) are concentrated in a very small region of the phase space (attractor) and their distances tend to zero. In spite of this, the invariant spectrum spreads to a finite region of values of a. This region, in comparison with the spectrum of the non-dissipative case in figure 6(b), shows a shift towards negative values of a.

In figure 7(a), another example of the Hénon map is given for a chaotic orbit leading to a strange attractor. The strange attractor appears for $K = 5.0$, $b = 0.5$ and is represented in figure 7(a) by a large number of lanes. The corresponding invariant spectrum is shown in figure 7(b). Again the full curve corresponds to the first million periods, starting at $x_0 = 0.1$, $y_0 = 0.5$, $y_{x0} = 0.0$, and the dots to the next million periods. The agreement is perfect.

It is remarkable that all types of dissipative orbits, e.g. orbits leading to a point attractor or to a strange attractor, have characteristic invariant spectra. We are working now on several extensions and applications of the invariant spectra. In particular, the role of the various well defined maxima of such spectra will be discussed in a future paper.

Acknowledgment

This research was supported in part by the EEC Human Capital and Mobility Program ERB4050 PL930312.

Note added in proof. We found recently some papers similar to ours. In particular Grassberger *et al* [5] (and references therein) introduce an 'effective Lyapunov exponent', while Froeschle *et al* [6] define a 'local Lyapunov indicator', which is essentially our 'stretching number'. However, the main result of our paper, namely the invariance of the spectra with respect to the initial time along every orbit, and with respect to the initial point in the (same) stochastic region, is new. A detailed comparison of these papers with ours will be given in a future publication.

References

[1] Contopoulos G, Galgani L and Giorgilli A 1978 *Phys. Rev.* A **18** 1183
[2] Udry S and Pfenniger O 1988 *Astron. Astrophys.* **198** 135
[3] Kandrup H E and Mahon M E 1994 *Three-Dimensional Dynamical Systems* ed S Gottesman *et al* (New York: Academic Sciences Annals) at press
[4] Hénon M 1976 *Commun. Math. Phys.* **50** 69
[5] Grassberger P, Badii R and Politi A 1988 *J. Stat. Phys.* **51** 135
[6] Froeschle C, Froeschle Ch and Lohinger E 1993 *Celest. Mech. Dyn. Astron.* **56** 307